

The Magnetohydrostatic Boundary Value Problem Without Symmetry

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To Professor Arnulf Schlüter on his 60th Birthday

The magnetohydrostatic boundary value problem is investigated for the case where the magnetic field is tangential on an toroidal surface without symmetry. The first part deals with small deviations from the homogeneous field. If the surface deviation from a cylinder with arbitrary cross-section is small of order ε , it is found that in first order a solution exists if two profile functions are properly prescribed. The solubility condition in the $(n+1)$ -th order determines the freedom of the n -th order. In the second part general toroidal geometry is considered for the case of a smooth pressure profile. The longitudinal current is determined so that the rotational transform vanishes.

Introduction

A magnetohydrostatic equilibrium is described by the equations

$$\left. \begin{aligned} \operatorname{div} \mathbf{B} &= 0, \\ \mathbf{j} \times \mathbf{B} - \nabla p &= 0, \\ \mathbf{j} &= \operatorname{curl} \mathbf{B}, \end{aligned} \right\} \quad (1)$$

where p is the hydrostatic pressure, \mathbf{B} the magnetic field, and \mathbf{j} the current density. Although, the system (1) of partial differential equations has been known for a long time, it is still controversial what additional specifications for system (1) will constitute a well-posed problem. This is because the system has a mathematically interesting property: it is of the mixed type, i.e. it possesses two real and two complex characteristics. It could perhaps be stated that the real characteristics belong to the “hyperbolic part” of the system, and the complex ones to the “elliptic part”.

The system (1) was first considered for the case of axisymmetry independently by Lüst and Schlüter [1], Grad and Rubin [2], and Shafranov [3] with the result that the system could be reduced to a second-order elliptic equation for which two so-called “profile functions” can be prescribed. For this case the boundary value problem that \mathbf{B} be tangential on an axisymmetric toroidal surface is well posed and a solution exists under certain conditions on the

profile functions. The boundary value problem is also well posed for helical symmetry [4], [5]. So, it might be conjectured that even without any symmetries the boundary value problem that \mathbf{B} be tangential on a toroidal surface is well posed if, in addition, two profile functions are prescribed. This, however, is controversial for the following reason. If one uses the so-called “ β -iteration”, which has already been discussed in [6], one is immediately confronted with the question whether a vacuum field which is tangential on a non-symmetric toroidal surface possesses so-called “magnetic surfaces”. This, however, is not a trivial question. Indeed, if the field lines of non-symmetric solutions of (1) are followed around the torus, usually so-called “islands” occur (see, for instance, [7]). So, it was concluded in [5] that, in general, smooth solutions of the system (1) do not exist. On the other hand, if the equilibrium has reflexional symmetry with respect to a poloidal plane, then a solution exists [8]. This is noteworthy because in this case the “hyperbolic part” and the “elliptic part” of the equations do not separate.

Nearly Symmetric Geometry

In the following, for a simple geometry the boundary value problem without any symmetry is investigated. Consider a cylinder with its axis in the direction \mathbf{e} and arbitrary cross-section G in the x, y plane. The homogeneous field

$$\mathbf{B}_0 = B \mathbf{e}, \quad B = \text{const}, \quad j_0 = 0$$

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is a solution of (1) with plane symmetry (which can be considered as a special case of helical symmetry). Let the perturbed boundary be periodic in z and the perturbation of the unperturbed boundary ∂G of G be of order ε . Let \hat{n} be the boundary normal of the perturbed domain and n the normal of ∂G . Then the expansion of the boundary and of the fields in the form

$$\begin{aligned} B &= B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \dots, \\ p &= \varepsilon p_1 + \varepsilon^2 p_2 + \dots \end{aligned}$$

leads to the result that the condition $\hat{n} \cdot B = 0$ on the perturbed boundary is equivalent to prescribing $n \cdot B$ on the unperturbed boundary ∂G with

$$n \cdot \overset{o}{B}_1 = 0 \text{ on } \partial G. \quad (2)$$

Here, the superscript o denotes the z -average.

In order that the straight geometry be equivalent to a torus ("topological torus"), all physical quantities have to be periodic in z with, say, period 2π . Let the Fourier expansion be written in the form

$$B_n = \sum_k \overset{k}{B}_n, \quad j_n = \sum_k \overset{k}{j}_n, \quad p_n = \sum_k \overset{k}{p}_n,$$

where the superscript k denotes the k -th Fourier component, which has the property

$$e \cdot \nabla \dots = i k \dots^{(k)}$$

In the n -th order Eq. (1) are

$$j_n \times B_0 - \nabla p_n = J_{n-1}, \quad \operatorname{div} B_n = 0, \quad (3)$$

$$j_n = \operatorname{curl} B_n, \quad n = 1, 2, 3, \dots,$$

$$J_{n-1} = B_1 \times j_{n-1} + B_2 \times j_{n-2} + \dots + \dots + B_{n-1} \times j_1,$$

$$n = 2, 3, 4, \dots,$$

$$J_0 = 0.$$

If the Cartesian components of $\overset{k}{B}_n$ and $\overset{k}{J}_n$ are

$$\overset{k}{B}_n = \begin{pmatrix} i f_n \\ i g_n \\ h_n \end{pmatrix}, \quad \overset{k}{J}_n = \begin{pmatrix} I_n \\ J_n \\ K_n \end{pmatrix},$$

$$n = 1, 2, 3, \dots,$$

then the system (3) is of the form

$$\left. \begin{aligned} \frac{\partial}{\partial x} \overset{k}{f}_n + \frac{\partial}{\partial y} \overset{k}{g}_n + k \overset{k}{h}_n &= 0, \\ -B \left(k \overset{k}{f}_n + \frac{\partial}{\partial x} \overset{k}{h}_n \right) - \frac{\partial}{\partial x} \overset{k}{p}_n &= I_{n-1}, \\ -B \left(k \overset{k}{g}_n + \frac{\partial}{\partial y} \overset{k}{h}_n \right) - \frac{\partial}{\partial y} \overset{k}{p}_n &= J_{n-1}, \\ -k \overset{k}{p}_n &= K_{n-1}, \quad n = 1, 2, 3, \dots \end{aligned} \right\} \quad (4)$$

For the solution of (4) two cases have to be distinguished:

1) $k = 0$, 2) $k \neq 0$.

1) $k = 0$. In this case the system is only solvable if the inhomogeneous terms satisfy the conditions

$$\begin{aligned} e \cdot \overset{o}{J}_n &= 0, \\ e \cdot \operatorname{curl} \overset{o}{J}_n &= 0, \quad n = 1, 2, 3, \dots \end{aligned} \quad (5)$$

If the conditions (5) are satisfied, the solution is

$$\begin{aligned} \overset{o}{B}_n &= h_n e - i e \times \nabla u_n, \\ \overset{o}{j}_n &= -e \times \nabla h_n - i e \Delta u_n, \\ \overset{o}{p}_n &= -F_{n-1} - B h_n, \quad n = 1, 2, 3, \dots, \end{aligned}$$

where

$$\begin{aligned} \overset{o}{J}_n &= \nabla F_n, \quad F_n(x, y), \quad n = 0, 1, 2, 3, \dots, \\ F_0 &= 0 \end{aligned}$$

is a particular solution of (5). The functions $h_n(x, y)$, $u_n(x, y)$ are determined by the higher-order solubility conditions.

2) $k \neq 0$: In this case the solution is

$$\left. \begin{aligned} \overset{k}{p}_n &= -\frac{\overset{k}{K}_{n-1}}{k}, \\ \overset{k}{f}_n &= -\frac{1}{k} \frac{\partial}{\partial x} \overset{k}{h}_n + \frac{1}{k^2 B} \frac{\partial \overset{k}{K}_{n-1}}{\partial x} - k \overset{k}{I}_{n-1}, \\ \overset{k}{g}_n &= -\frac{1}{k} \frac{\partial}{\partial y} \overset{k}{h}_n + \frac{1}{k^2 B} \frac{\partial \overset{k}{K}_{n-1}}{\partial y} - k \overset{k}{J}_{n-1}, \end{aligned} \right\} \quad (6)$$

$$n = 1, 2, 3, \dots,$$

where the function $h_n(x, y)$ has a given normal derivative on ∂G and satisfies

$$\Delta h_n = \frac{1}{B} \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{k}{k} K_{n-1} - \frac{\partial I_{n-1}}{\partial x} - \frac{\partial J_{n-1}}{\partial y} \right],$$

$$n = 1, 2, 3, \dots \quad (7)$$

In vector notation (7) is of the form

$$\Delta \mathbf{B}_n \cdot \mathbf{e} = \frac{i}{kB} (\mathbf{e} \cdot \nabla \operatorname{div} \mathbf{J}_{n-1} - \Delta \mathbf{e} \cdot \mathbf{J}_{n-1}),$$

while (6) can be expressed in the form

$$\mathbf{j}_n = -\frac{i}{kB} \operatorname{curl} \mathbf{J}_{n-1}, \quad n = 1, 2, 3, \dots \quad (8)$$

With these preparations one is able to discuss explicitly the lower orders. Because of $\mathbf{j}_1 = 0$ for $k \neq 0$ it is found that

$$\begin{aligned} \overset{o}{\mathbf{J}}_1 &= \overset{o}{\mathbf{B}}_1 \times \overset{o}{\mathbf{j}}_1 \\ &= \nabla \frac{h_1^2}{2} - (\nabla u) \Delta u - i \mathbf{e} \mathbf{e} \cdot (\nabla h_1 \times \nabla u), \\ u &= u_1. \end{aligned}$$

This yields the second order solubility conditions

$$\begin{aligned} \mathbf{e} \cdot (\nabla h_1 \times \nabla u) &= 0, \\ \mathbf{e} \cdot (\nabla u \times \nabla \Delta u) &= 0, \end{aligned}$$

which can be satisfied by choosing

$$h_1 = H(u), \quad \Delta u = U'(u). \quad (9)$$

Because of (2) the boundary condition for (9) is $u = \text{const}$ on ∂G . $H(u)$ and $U'(u)$ are the two profile functions which can be prescribed. If the function $U'(u)$ and the boundary ∂G are chosen properly, then (9) together with the boundary condition $u = \text{const}$ on ∂G represents a well-posed problem and a solution exists (see, for instance, [9]). Thus, it is found that

$$\begin{aligned} \overset{o}{\mathbf{B}}_1 &= H \mathbf{e} - i \mathbf{e} \times \nabla u, \\ \mathbf{j}_1 &= -\mathbf{e} \times \nabla H - i \mathbf{e} \Delta u \end{aligned}$$

and

$$\begin{aligned} \mathbf{J}_1 &= \mathbf{B}_1 \times \mathbf{j}_1 \\ &= \mathbf{B}_1 \cdot \mathbf{e} \nabla H - \mathbf{e} (\mathbf{B}_1 \cdot \nabla H) + i \mathbf{e} \times \mathbf{B}_1 \Delta u, \end{aligned} \quad (10)$$

$$\begin{aligned} \operatorname{curl} \mathbf{J}_1 &= -(\nabla H) \times \nabla (\mathbf{B}_1 \cdot \mathbf{e}) + \mathbf{e} \times \nabla (\mathbf{B}_1 \cdot \nabla H) \\ &\quad + k \mathbf{B}_1 \Delta u + i \mathbf{e} \mathbf{B}_1 \cdot \nabla \Delta u, \end{aligned} \quad (11)$$

$$F_1 = \frac{1}{2} H^2 - U.$$

Deriving the third-order solubility conditions is more tedious and will be done in the Appendix. The result is

$$0 = \mathbf{e} \cdot \overset{o}{\mathbf{J}}_2 = -i (\mathbf{e} \times \nabla u) \quad (12)$$

$$\cdot \nabla \left[H' u_2 - h_2 + \frac{H'}{2B} \sum_{k \neq 0} \frac{1}{k} \mathbf{e} \cdot (\mathbf{B}_1 \times \mathbf{B}_1) \right],$$

$$0 = \mathbf{e} \cdot \operatorname{curl} \overset{o}{\mathbf{J}}_2 = (\mathbf{e} \times \nabla u) \quad (13)$$

$$\cdot \nabla \left\{ \Delta u_2 - U'' u_2 + \frac{1}{B} \sum_{k \neq 0} \left[H' \mathbf{B}_1 \cdot \mathbf{B}_1 - \frac{U''}{2k} \mathbf{e} \cdot (\mathbf{B}_1 \times \mathbf{B}_1) \right] \right\}.$$

Equations (12), (13) determine the functions $u_2(x, y)$, $h_2(x, y)$. Since these equations contain the operator $(\mathbf{e} \times \nabla u) \cdot \nabla$, which differentiates along the closed lines $u = \text{const}$, and since u_2, h_2 are single-valued, it is important to note that the operator also acts on the inhomogeneous terms in the equations. So, (12), (13) can be integrated by setting the expression in brackets of (12) and the expression in braces of (13) equal to zero:

$$H' u_2 - h_2 + \frac{H'}{2B} \sum_{k \neq 0} \frac{\mathbf{e}}{k} \cdot (\mathbf{B}_1 \times \mathbf{B}_1) = 0, \quad (14)$$

$$\begin{aligned} \Delta u_2 - U'' u_2 \\ + \frac{1}{B} \sum_{k \neq 0} \left[\mathbf{B}_1 \cdot \mathbf{B}_1 - \frac{U''}{2k} \mathbf{e} \cdot (\mathbf{B}_1 \times \mathbf{B}_1) \right] = 0. \end{aligned} \quad (15)$$

Equation (15) together with the condition that u_2 be given on the boundary ∂G represents a well-posed linear, inhomogeneous, elliptic problem which has a solution if $-U'' < \lambda$, where λ is the lowest eigenvalue of the problem $\Delta u_2 + \lambda u_2 = 0$, $u_2 = 0$ on ∂G . After u_2 has been determined from (15), h_2 is given algebraically by (14).

In higher order the same structure is found: all terms of $\mathbf{e} \cdot \overset{o}{\mathbf{J}}_n$, $\mathbf{e} \cdot \operatorname{curl} \overset{o}{\mathbf{J}}_n$ contain the operator $(\mathbf{e} \times \nabla u) \cdot \nabla$. Thus, u_n and h_n can be determined as single-valued functions. If the profile functions U', H are analytical, then the solution exists in all

orders. However, the convergence of the series is still an open question.

More far-reaching statements can be made for the vacuum case $U' \equiv H \equiv 0$. In this case the problem reduces to the question whether magnetic surfaces exist if the boundary value problem is solved. It will be shown in a forthcoming paper that these surfaces indeed exist.

Vanishing rotational transform

If the configuration has reflexional symmetry with respect to a poloidal plane such that \mathbf{B} is normal and \mathbf{j} is tangential on the plane of symmetry, then the rotational transform vanishes. If, in addition, \mathbf{B} is tangential on a toroidal surface with the respective symmetry then the so-called " β -iteration" can be applied [8] to construct a solution. The question thus arises whether it is possible to find an iteration which keeps the rotational transform identically zero, even if the reflexional symmetry is disturbed.

Let ∂T be a smooth toroidal surface without any symmetry. Let $\psi(\mathbf{r}) = \text{const}$, $\chi(\mathbf{r}) = \text{const}$ be the closed field lines of a field \mathbf{B}_n which nowhere vanishes inside the toroid and which is tangential on ∂T . The well-known representation

$$\mathbf{B}_n = \nabla\psi \times \nabla\chi \quad (16)$$

implies that

$$\text{div } \mathbf{B}_n = 0.$$

The functions $\psi(\mathbf{r})$, $\chi(\mathbf{r})$ are well-suited to serve as poloidal coordinates. In addition, a toroidal coordinate function $\sigma(\mathbf{r})$ is chosen which increases monotonically along the field lines of \mathbf{B}_n with total increase 1. Furthermore, let the functional determinant

$$D = (\nabla\psi \times \nabla\chi) \cdot \nabla\sigma$$

be everywhere positive.

In order to construct a new field \mathbf{B}_{n+1} , the equation

$$\mathbf{j}_{n+1} \times \mathbf{B}_n = \nabla p_n \quad (17)$$

is solved algebraically to yield

$$\mathbf{j}_{n+1} = \bar{\gamma} \mathbf{B}_n + \mathbf{J}, \quad \bar{\gamma}(\psi, \chi), \quad (18)$$

where

$$\bar{\gamma} = \int_0^1 \dots d\sigma, \quad \mathbf{J} = \tilde{\gamma} \mathbf{B}_n + B_n^{-2} \mathbf{B}_n \times \nabla p_n, \quad \tilde{\gamma} = 0.$$

The function $\tilde{\gamma}$ is determined from the solenoidal property of \mathbf{J}

$$0 = \text{div } \mathbf{J} = \mathbf{B}_n \cdot \nabla \tilde{\gamma} + \text{div } B_n^{-2} \mathbf{B}_n \times \nabla p_n. \quad (19)$$

If the coordinates ψ, χ are chosen such that $p_n(\psi)$, then by introducing the covariant components by

$$\mathbf{B}_n = B_\psi \nabla\psi + B_\chi \nabla\chi + B_\sigma \nabla\sigma,$$

$$B_\psi = \mathbf{B}_n \cdot \frac{\partial \mathbf{r}}{\partial \psi}, \quad B_\chi = \mathbf{B}_n \cdot \frac{\partial \mathbf{r}}{\partial \chi},$$

$$B_\sigma = \mathbf{B}_n \cdot \frac{\partial \mathbf{r}}{\partial \sigma}$$

it is found that

$$\begin{aligned} \text{div } B_n^{-2} \mathbf{B}_n \times \nabla p_n &= p'_n \text{div } B_n^{-2} \mathbf{B}_n \times \nabla\psi \\ &= p'_n D \left(\frac{\partial}{\partial \chi} \frac{B_\sigma}{B_n^2} - \frac{\partial}{\partial \sigma} \frac{B_\chi}{B_n^2} \right). \end{aligned}$$

Because of $\mathbf{B}_n \cdot \nabla = D \partial / \partial \sigma$ Eq. (19) acquires the form

$$\frac{\partial}{\partial \sigma} \tilde{\gamma} + p'_n \left(\frac{\partial}{\partial \chi} \frac{B_\sigma}{B_n^2} - \frac{\partial}{\partial \sigma} \frac{B_\chi}{B_n^2} \right) = 0. \quad (20)$$

Equation (20) is solvable if and only if [10]

$$0 = p'_n \frac{\partial}{\partial \chi} \int_0^1 \frac{B_\sigma}{B_n^2} d\sigma,$$

which means that the surfaces of constant p_n must coincide with the surfaces where the quantity

$$\begin{aligned} q &= \overline{B_n^{-2} B_\sigma} = \overline{B_n^{-2} \mathbf{B}_n \cdot \frac{\partial \mathbf{r}}{\partial \sigma}} \\ &= \oint B_n^{-2} \mathbf{B}_n \cdot d\mathbf{r} = \oint B_n^{-1} dl \end{aligned}$$

is constant.

From the current density (18) a new field

$$\mathbf{b} = \tilde{\mathbf{B}} + \mathbf{B}^*$$

is determined, where the fields $\tilde{\mathbf{B}}$, \mathbf{B}^* satisfy the equations

$$\left. \begin{aligned} \text{curl } \tilde{\mathbf{B}} &= \mathbf{J}, \\ \text{div } \tilde{\mathbf{B}} &= 0, \\ \mathbf{n} \cdot \tilde{\mathbf{B}} &= 0 \text{ on } \partial T, \\ \oint \tilde{\mathbf{B}} \cdot d\mathbf{r} &= 0; \end{aligned} \right\} \quad (21)$$

$$\left. \begin{aligned} \text{curl } \mathbf{B}^* &= \bar{\gamma} \mathbf{B}_n, \\ \text{div } \mathbf{B}^* &= 0, \\ \mathbf{n} \cdot \mathbf{B}^* &= 0 \text{ on } \partial T, \\ \oint \mathbf{B}^* \cdot d\mathbf{r} &= I. \end{aligned} \right\} \quad (22)$$

The line integrals in (21), (22) are taken along a curve $\psi = \text{const}$, $\chi = \text{const}$ on ∂T . Thus, I is the total current flowing on the main axis of the toroid. According to Kress [11] Eqs. (21), (22) represent wellposed elliptical problems and a solution exists if the r.h. sides satisfy certain smoothness conditions.

In general, the field \mathbf{b} will have a rotational transform. First, it will be shown that it is possible to choose the function $\bar{\gamma}(\psi, \chi)$ such that the flux

$$\varphi = \iint \mathbf{b} \cdot d^2 f$$

through a ribbon bounded by any two lines of \mathbf{B}_n vanishes. Since the surface elements on the surfaces $\psi = \text{const}$, $\chi = \text{const}$ are

$$d^2 f = D^{-1} \nabla \psi d\chi d\sigma,$$

$$d^2 f = D^{-1} \nabla \chi d\psi d\sigma,$$

respectively, the flux φ can be expressed by the contravariant components of \mathbf{b} , which are defined by

$$\mathbf{b} = b^\psi \frac{\partial \mathbf{r}}{\partial \psi} + b^\chi \frac{\partial \mathbf{r}}{\partial \chi} + b^\sigma \frac{\partial \mathbf{r}}{\partial \sigma},$$

$$b^\psi = \mathbf{b} \cdot \nabla \psi, \quad b^\chi = \mathbf{b} \cdot \nabla \chi, \quad b^\sigma = \mathbf{b} \cdot \nabla \sigma.$$

In terms of these one has

$$\varphi = \int_{\psi_0}^{\psi} \int_0^1 \frac{b^\chi}{D} d\psi d\sigma + \int_{\chi_0}^{\chi} \int_0^1 \frac{b^\psi}{D} d\chi d\sigma,$$

where $\psi_0, \chi_0 = \text{const}$ are field lines on ∂T . The introduction of a function $\Gamma(\psi, \chi)$ by

$$\bar{\gamma} = \frac{\partial \Gamma}{\partial \psi}, \quad \bar{\gamma} \mathbf{B}_n = \nabla \Gamma \times \nabla \chi$$

allows integration of the first of Eqs. (22) in the form

$$\mathbf{B}^* = \Gamma \nabla \chi + \nabla W,$$

where W has to be determined by the rest of (22):

$$\begin{aligned} \operatorname{div}(\Gamma \nabla \chi) + \Delta W &= 0, \\ \mathbf{n} \cdot \nabla W &= -\Gamma \mathbf{n} \cdot \nabla \chi \text{ on } \partial T, \\ W(\sigma = 1) - W(\sigma = 0) &= I. \end{aligned} \quad (23)$$

The condition for the vanishing of the flux φ now is

$$\begin{aligned} \Gamma D^{-1} (\nabla \chi)^2 + \overline{D^{-1} \nabla W \cdot \nabla \chi} \\ + \overline{D^{-1} \tilde{B} \chi} = 0 \end{aligned} \quad (24)$$

because if

$$\partial \varphi / \partial \psi = \overline{D^{-1} b^\chi} = 0,$$

then φ is a function of χ alone. However, such a function must be zero because φ vanishes on ∂T . Equations (23), (24) can be solved iteratively by the scheme

$$\Gamma_m \Rightarrow W_m [\text{from (23)}] \Rightarrow \Gamma_{m+1} [\text{from (24)}].$$

After this procedure has converged the conditions

$$\overline{D^{-1} b^\chi} = \overline{D^{-1} b^\psi} = 0 \quad (25)$$

are satisfied, i.e. φ vanishes.

Next, the field line equations for the vector field \mathbf{b} are considered:

$$d\mathbf{r}/d\sigma = \mathbf{b}/b^\sigma$$

or, with ψ, χ as dependent variables,

$$\frac{d\psi}{d\sigma} = \frac{b^\psi}{b^\sigma}, \quad \frac{d\chi}{d\sigma} = \frac{b^\chi}{b^\sigma}. \quad (26)$$

The initial conditions for (26) are

$$\psi(\psi_0, \chi_0, 0) = \psi_0, \quad \chi(\psi_0, \chi_0, 0) = \chi_0.$$

Although the flux condition $\varphi \equiv 0$ is satisfied, the solutions of (26), in general, have a rotational transform. However, this rotational transform is of order ε^2 , where

$$\varepsilon = \max_r |\mathbf{b} - \mathbf{B}_n|,$$

as was shown in, for instance, [12]. Thus, the displacements

$$\delta\psi = \psi(\psi_0, \chi_0, 1) - \psi_0,$$

$$\delta\chi = \chi(\psi_0, \chi_0, 1) - \chi_0$$

are also of order ε^2 . The new flux functions

$$\psi' = \psi(\psi_0, \chi_0, \sigma) - \sigma \delta\psi(\psi_0, \chi_0),$$

$$\chi' = \chi(\psi_0, \chi_0, \sigma) - \sigma \delta\chi(\psi_0, \chi_0)$$

are then periodic in σ , and the new field

$$\mathbf{B}_{n+1} = \nabla \psi' \times \nabla \chi', \quad (27)$$

with

$$\psi', \chi'(\psi_0(\mathbf{r}), \chi_0(\mathbf{r}), \sigma(\mathbf{r})),$$

has closed field lines. The new field (27) is also tangential on ∂T as is seen by the following argument. One can choose a gauge such that $\psi = \text{const}$ on ∂T . Then, because \mathbf{b} is tangential on ∂T , $\delta\psi = 0$

on ∂T . So, $\psi' = \text{const}$ on ∂T . So, \mathbf{B}_{n+1} is tangential on ∂T .

The question of convergence of the scheme has not been discussed. However, the procedure is similar to the symmetric case discussed in [8], where convergence can be shown. The main difference to the present case is the introduction of flux functions ψ , χ , and the question is whether this is really necessary.

Configurations without rotational transform still deserve further consideration because it is conjectured that these configurations do not show any

resistive instabilities. So it might be that if these configurations are ideally stable, then they are also stable with respect to resistive modes.

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Appendix

The formulae

$$\overset{o}{\mathbf{B}}_2 = h_2 \mathbf{e} - i \mathbf{e} \times \nabla u_2, \quad \overset{o}{\mathbf{j}}_2 = -\mathbf{e} \times \nabla h_2 - i \mathbf{e} \Delta u_2,$$

$$\overset{o}{\mathbf{B}}_1 = H \mathbf{e} - i \mathbf{e} \times \nabla u, \quad \overset{o}{\mathbf{j}}_1 = -\mathbf{e} \times \nabla H - i \mathbf{e} \Delta u,$$

$$\overset{k}{\mathbf{j}}_2 = -\frac{i}{kB} \text{curl } \overset{k}{\mathbf{J}}_1 = -\frac{i}{kB} [-(\nabla H) \times \nabla (\overset{k}{\mathbf{B}}_1 \cdot \mathbf{e}) + \mathbf{e} \times \nabla (\overset{k}{\mathbf{B}}_1 \cdot \nabla H) + k \overset{k}{\mathbf{B}}_1 \Delta u + i \mathbf{e} (\overset{k}{\mathbf{B}}_1 \cdot \nabla \Delta u)], \quad k \neq 0$$

are used to compute

$$\begin{aligned} \overset{o}{\mathbf{J}}_2 &= \sum_k \overset{k}{\mathbf{B}}_1 \times \overset{-k}{\mathbf{j}}_2 + \overset{k}{\mathbf{B}}_2 \times \overset{-k}{\mathbf{j}}_1 = \overset{o}{\mathbf{B}}_2 \times \overset{o}{\mathbf{j}}_1 + \overset{o}{\mathbf{B}}_1 \times \overset{o}{\mathbf{j}}_2 + \sum_{k \neq 0} \overset{k}{\mathbf{B}}_1 \times \overset{-k}{\mathbf{j}}_2 \\ &= h_2 \mathbf{e} \times (-\mathbf{e} \times \nabla H - i \mathbf{e} \Delta u) - i (\mathbf{e} \times \nabla u_2) \times (-\mathbf{e} \times \nabla H - i \mathbf{e} \Delta u) + H \mathbf{e} \times (-\mathbf{e} \times \nabla h_2 - i \mathbf{e} \Delta u_2) \\ &\quad - i (\mathbf{e} \times \nabla u) \times (-\mathbf{e} \times \nabla h_2 - i \mathbf{e} \Delta u_2) + \sum_{k \neq 0} \frac{i}{kB} \overset{k}{\mathbf{B}}_1 \times \text{curl } \overset{-k}{\mathbf{J}}_1 \\ &= h_2 \nabla H - (\nabla u_2) \Delta u - i \mathbf{e} (\nabla H \times \nabla u_2) \cdot \mathbf{e} + H \nabla h_2 - i \mathbf{e} (\nabla h_2 \times \nabla u) \cdot \mathbf{e} - (\nabla u) \Delta u_2 \\ &\quad + \frac{i}{B} \sum_{k \neq 0} \frac{1}{k} \{ -\overset{k}{\mathbf{B}}_1 \times [(\nabla H) \times \nabla (\overset{-k}{\mathbf{B}}_1 \cdot \mathbf{e})] \\ &\quad + \overset{k}{\mathbf{B}}_1 \times [\mathbf{e} \times \nabla (\overset{-k}{\mathbf{B}}_1 \cdot \nabla H)] - k \overset{k}{\mathbf{B}}_1 \times \overset{-k}{\mathbf{B}}_1 \Delta u + i \overset{k}{\mathbf{B}}_1 \times \mathbf{e} (\overset{-k}{\mathbf{B}}_1 \cdot \nabla \Delta u) \} \\ &= \nabla h_2 H - U' \nabla u_2 - (\nabla u) \Delta u_2 + \frac{i}{B} \sum_{k \neq 0} \frac{1}{k} \{ -(\nabla H) \overset{k}{\mathbf{B}}_1 \cdot \nabla (\overset{-k}{\mathbf{B}}_1 \cdot \mathbf{e}) + \nabla [(\overset{-k}{\mathbf{B}}_1 \cdot \mathbf{e}) (\overset{k}{\mathbf{B}}_1 \cdot \nabla H)] \\ &\quad + i \overset{k}{\mathbf{B}}_1 \times \mathbf{e} (\overset{-k}{\mathbf{B}}_1 \cdot \nabla U') \} + \mathbf{e} \left[-i (\mathbf{e} \times \nabla u) \cdot \nabla (H' u_2 - h_2) + \frac{i}{B} \sum_{k \neq 0} \frac{1}{k} \overset{k}{\mathbf{B}}_1 \cdot \nabla (\overset{-k}{\mathbf{B}}_1 \cdot \nabla H) \right]. \end{aligned}$$

In the last expression the first two lines represent the poloidal part, and the last line the toroidal part of $\overset{o}{\mathbf{J}}_2$. The last sum can be written in the alternative form

$$\begin{aligned} \sum_{k \neq 0} \frac{1}{k} \overset{k}{\mathbf{B}}_1 \cdot \nabla (\overset{-k}{\mathbf{B}}_1 \cdot \nabla H) &= -(\mathbf{e} \times \nabla H) \cdot \nabla \sum_{k \neq 0} \frac{1}{k} f_1 g_1 = -(\mathbf{e} \times \nabla u) \cdot \nabla H' \sum_{k \neq 0} \frac{1}{k} f_1 g_1 \\ &= -\frac{1}{2} (\mathbf{e} \times \nabla u) \cdot \nabla H' \sum_{k \neq 0} \frac{1}{k} \mathbf{e} \cdot (\overset{-k}{\mathbf{B}}_1 \times \overset{k}{\mathbf{B}}_1), \end{aligned} \quad (\text{A1})$$

where the first of (4) has been used. The toroidal part of $\overset{o}{J}_2$ is thus

$$\mathbf{e} \cdot \overset{o}{J}_2 = -i(\mathbf{e} \times \nabla u) \cdot \nabla \left[H' u_2 - h_2 + \frac{H'}{2B} \sum_{k \neq 0} \frac{\mathbf{e}}{k} \cdot (\mathbf{B}_1^{\bar{k}} \times \mathbf{B}_1^k) \right], \quad (\text{A2})$$

and for the toroidal part of $\text{curl } \overset{o}{J}_2$ it is found that (A3)

$$\mathbf{e} \cdot \text{curl } \overset{o}{J}_2 = (\mathbf{e} \times \nabla u) \cdot \nabla \left[\Delta u_2 - U'' u_2 + H' \frac{i}{B} \sum_{k \neq 0} \frac{1}{k} \mathbf{B}_1^k \cdot \nabla (\mathbf{B}_1^{\bar{k}} \cdot \mathbf{e}) \right] + \frac{U''}{B} \sum_{k \neq 0} \frac{1}{k} \mathbf{B}_1^k \cdot \nabla (\mathbf{B}_1^{\bar{k}} \cdot \nabla u).$$

Here, the first sum yields

$$\sum_{k \neq 0} \frac{1}{k} \mathbf{B}_1^k \cdot \nabla (\mathbf{B}_1^{\bar{k}} \cdot \mathbf{e}) = -i \sum_{k \neq 0} \mathbf{B}_1^k \cdot \mathbf{B}_1^{\bar{k}},$$

where (6) have been used. Since the last sum in (A3) is similar to (A1), the result is

$$\mathbf{e} \cdot \text{curl } \overset{o}{J}_2 = (\mathbf{e} \times \nabla u) \cdot \nabla \left\{ \Delta u_2 - U'' u_2 + \frac{1}{B} \sum_{k \neq 0} \left[H' \mathbf{B}_1^k \cdot \mathbf{B}_1^{\bar{k}} - \frac{U''}{2k} \mathbf{e} \cdot (\mathbf{B}_1^{\bar{k}} \times \mathbf{B}_1^k) \right] \right\}. \quad (\text{A4})$$

Equations (A2) and (A4) are (12) and (13) of the main text.

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